On additive preservers of Drazin invertibility

Mourad Oudghiri
(joint work with M. Mbekhta and K. Souilah)

Université Mohammed I, Oujda, Maroc

Séminaire d’Analyse Fonctionnelle
Laboratoire LAGA
13 mai 2016 - Oujda
Notations:

- $X$: Infinite-dimensional complex Banach space.
- $X^*$: Topological dual space of $X$.
- $\mathcal{L}(X)$: The algebra of all bounded linear operators acting on $X$.
- $N(T)$: Kernel of $T \in \mathcal{L}(X)$.
- $R(T)$: Range of $T \in \mathcal{L}(X)$.

Additive map $U : X \rightarrow Y$ means $U(x + y) = Ux + Uy$.

Conjugate linear map $U$ is additive and $U(\lambda x) = \overline{\lambda} Ux$.

$\Phi$: Additive surjective map on $\mathcal{L}(X)$.

Question:

Let $S \subseteq \mathcal{L}(X)$. Characterize those surjective linear, or additive, maps on $\mathcal{L}(X)$ which preserves $S$ in both directions, that is:

$$T \in S \iff \Phi(T) \in S.$$
The ascent \(a(T)\) and descent \(d(T)\) of \(T \in \mathcal{L}(X)\) are defined by

\[
\begin{align*}
a(T) & = \inf\{ n \geq 0 : N(T^n) = N(T^{n+1}) \} \\
d(T) & = \inf\{ n \geq 0 : R(T^n) = R(T^{n+1}) \},
\end{align*}
\]

where \(\inf\emptyset = +\infty\).

\(T \in \mathcal{L}(X)\) is said to be **Drazin invertible**, if there exists \(S \in \mathcal{L}(X)\) and \(n \in \mathbb{N}\) such that

\[
STS = S, \quad TS = ST \quad \text{and} \quad T^{n+1}S = T^n.
\]

- \(S\) is **unique** and it is called **Drazin inverse** of \(T\).
- The smallest non-negative integer \(n\) is denoted by \(i(T)\).
- \(T\) is **Drazin invertible** if and only if \(\max\{\text{asc}(T), d(T)\} < \infty\). In this case, we have \(a(T) = d(T) = i(T)\).
- \(T\) is Drazin invertible \(\iff T^*\) is Drazin invertible, and in this case, \(i(T^*) = i(T)\).
An operator $T \in \mathcal{L}(X)$ is called *Fredholm* if

$$\max\{\dim N(T), \text{codim } R(T)\} < \infty.$$ 

The range of such operator $T$ is automatically closed.

- $\mathcal{D}^r(X)$: the set of Drazin invertible operators.
- $\mathcal{B}(X)$: the set of *Browder* operators, i.e Fredholm operators which are Drazin invertible.
Let $H$ be a separable infinite-dimensional Hilbert space, let $\phi : \mathcal{L}(H) \to \mathcal{L}(H)$ be a surjective additive continuous map. Then the following assertions are equivalent:

1. $\phi$ preserves $\mathcal{D}^r$;
2. $\phi$ preserves $\mathcal{B}$;
3. there exists an invertible bounded linear, or conjugate linear, operator $A : H \to H$ and a non-zero complex number $c$ such that either

$$\phi(S) = cASA^{-1} \text{ or } \phi(S) = cAS^*A^{-1}.$$ 

**Question**

Do the previous theorems remain true if we omit the continuity of $\Phi$ or the separability of $H$?
For $n \geq 0$:
\[ \mathcal{D}_n^r(X) = \{ T \in \mathcal{L}(X) : T \text{ is Drazin invertible and } i(T) \leq n \}.\]

**Question**

Given an integer $n \geq 0$. Characterize those additive maps on $\mathcal{L}(X)$ which preserve $\mathcal{D}_n^r(X)$ in both directions.

**Special cases:**

1. $n = 0$ : Jafarian-Sourour in 1986, and Šemrl in 2001
2. $n = 1$ : Mbekhta-Oudghiri in 2014.
Notations:

- **Hyper-kernel of** $T : \mathcal{N}^\infty(T) = \bigcup \mathcal{N}(T^n)$.
- **Hyper-range of** $T : \mathcal{R}^\infty(T) = \bigcap \mathcal{R}(T^n)$.
- $\mathcal{B}_n(X) = \{ T \in \mathcal{L}(X) : \text{Browder and } \dim \mathcal{N}^\infty(T) \leq n \}$.

Remarks:

1. $\mathcal{B}_n(X) \subseteq \mathcal{D}^r_n(X)$
2. For $T \in \mathcal{D}(X)$:

   $$X = \mathcal{N}^\infty(T) \oplus \mathcal{R}^\infty(T) \text{ and } T|_{\mathcal{R}^\infty(T)} \text{ is invertible}.$$  

3. $T \in \mathcal{B}_n(X)$ if and only if $T^* \in \mathcal{B}_n(X^*)$
Let \( \Phi : \mathcal{L}(X) \to \mathcal{L}(X) \) be a surjective additive map. The following assertions are equivalent.

(i) \( \phi \) preserves \( \mathcal{D}_n^r(X) \) in both direction;

(ii) \( \phi \) preserves \( \mathcal{B}_n(X) \);

(iii) there exist a scalar \( \alpha \neq 0 \) and a bounded invertible linear, or conjugate linear, operator \( A \) such that either

\[
\Phi(T) = \alpha ATA^{-1} \quad \text{or} \quad \Phi(T) = \alpha AT^* A^{-1}.
\]

The second form in assertion (iii) can occur if and only if \( X \) is reflexive.

Indeed, for each \( S \in \mathcal{L}(X^*) \), there exists \( T \in \mathcal{L}(X) \) such that \( S = T^* \).
Consequences

For $n = 0 : \mathcal{D}_0^r(X)$ is the set of invertible operators.

**Theorem (Jafarian-Sourour 1986, Šemrl 2001)**

Let $\Phi : \mathcal{L}(X) \rightarrow \mathcal{L}(X)$ be a surjective additive map. The following assertions are equivalent.

(i) $\Phi$ preserves the set of invertible operators in both directions;

(ii) there exist a scalar $\alpha \neq 0$ and a bounded invertible linear, or conjugate linear, operator $A$ such that either

$$\Phi(T) = \alpha ATA^{-1} \text{ or } \Phi(T) = \alpha AT^*A^{-1}.$$
Consequences

For $n = 1$ : $\mathcal{D}_{1}^{r}(X)$ is the set of group invertible operators

**Theorem (Mbekhta-Oudghiri 2014)**

Let $\Phi : \mathcal{L}(X) \to \mathcal{L}(X)$ be a surjective additive map. The following assertions are equivalent.

- (i) $\Phi$ preserves the set of group invertible operators in both directions;
- (ii) there exist a scalar $\alpha \neq 0$ and a bounded invertible linear, or conjugate linear, operator $A$ such that either

$$\Phi(T) = \alpha ATA^{-1} \text{ or } \Phi(T) = \alpha AT^{*}A^{-1}.$$
Proof of main theorem
Theorem

Let \( T \in \mathcal{L}(X) \). The following assertions are equivalent:

(i) \( T \in \mathcal{B}_n(X) \),

(ii) for every \( S \in \mathcal{L}(X) \) there exists \( \varepsilon_0 > 0 \) such that \( T + \varepsilon S \in \mathcal{D}_n^r(X) \) for all number (equivalently rational number) \( \varepsilon < \varepsilon_0 \).

Consequences

1. \( \Phi \) preserves \( \mathcal{D}_n^r(X) \) \( \Rightarrow \) \( \Phi \) preserves \( \mathcal{B}_n(X) \).
2. \( \mathcal{B}_n(X) \) is the topological interior of \( \mathcal{D}_n^r(X) \).
Proposition

Let $F \in \mathcal{L}(X)$ be a non-zero operator. Then

(i) there exists $T \in \mathcal{L}(X)$ invertible such that $T + F \notin \mathcal{B}_n(X)$;

(ii) if $\dim R(F) \geq 2$, there exists $T \in \mathcal{L}(X)$ invertible such that $T + F \notin \mathcal{B}_n(X)$ and $T - F \notin \mathcal{B}_n(X)$.

Consequence

If $\Phi$ preserves $\mathcal{B}_n(X)$ then $\Phi$ is injective.

Proof.
Suppose that $\Phi(F) = 0$ where $F \neq 0$. Let $T$ be invertible such that $T + F \notin \mathcal{B}_n(X)$. Then

$$\Phi(T + F) = \Phi(T) \in \mathcal{B}_n(X),$$

and so $T + F \in \mathcal{B}_n(X)$, the desired contradiction.
Theorem

Let $T \in \mathcal{B}_n(X)$, $z \in X$ and $f \in X^*$, and put $p = n - \dim \mathcal{N}^\infty(T)$. Then:

1. $z = a + b$ where $a \in \mathcal{N}^\infty(T)$ and $b \in \mathcal{R}^\infty(T)$,
2. $T + z \otimes f \notin \mathcal{B}_n(X)$ if and only if:
   - $f(T^i a) = 0$ for all $i \geq 0$;
   - $f(T_{o i}^{-i} b) = -\delta_{i1}$ for $1 \leq i \leq p + 1$ where $T_{o i} = T|_{\mathcal{R}^\infty(T)}$.

Corollary

Let $F$ be nonzero. The following assertions are equivalent:

1. $F$ is rank one;
2. For every $T \in \mathcal{B}_n(X)$, $\text{Card}\{\lambda \in \mathbb{Q} : T + \lambda F \notin \mathcal{B}_n(X)\} \leq 1$.

Consequence

If $\Phi$ preserves $\mathcal{G}_o$ then $\Phi$ preserves the set of rank 1 operators.
Proposition

If $\Phi : \mathcal{L}(X) \to \mathcal{L}(X)$ preserves $\mathcal{B}_n(X)$, then

1. there exists $\alpha \in \mathbb{C}^*$ such that $\Phi(l) = \alpha l$,
2. there exists an invertible bounded linear, or conjugate linear, operator $A$ such that

$$\Phi(F) = \alpha AFA^{-1} \text{ or } \Phi(F) = \alpha AF^*A^{-1},$$

for all finite rank operator $F$.

Proposition

Let $T, S \in \mathcal{L}(X)$ be two invertible operators such that

$$T + F \in \mathcal{B}_n(X) \iff S + F \in \mathcal{B}_n(X)$$

for every finite rank operator $F \in \mathcal{L}(X)$. Then $T = S$. 
Proof of theorem

Suppose that $\Phi$ preserves $G_0$ and $\Phi(F) = \alpha UFU^{-1}$ for all finite rank $F$. Then

$$\psi(T) = \alpha^{-1} U^{-1} TU$$

satisfies the same hypothesis as $\Phi$. Furthermore, $\psi(I) = I$ and $\psi(F) = F$ for all finite rank $F$. Let $\lambda \in \mathbb{Q} \setminus (\sigma(T) \cup \sigma(\psi(T)))$. We have

$$T + F \in G_0(X) \iff \psi(T) + F \in G_0(X),$$

for all finite rank $F$. Hence, $\psi(T) = T$.

The same argument can be used for the second form.
Remark

The main theorem can be without any change formulated for additive surjective mappings $\Phi : \mathcal{L}(X) \to \mathcal{L}(Y)$.

M. Mbekhta and M. Oudghiri [Preprint 2014]

Let $\Phi : \mathcal{L}(X) \to \mathcal{L}(Y)$ be a surjective additive map. The following assertions are equivalent.

(i) $\Phi$ preserves $D_n^r$;
(ii) $\Phi$ preserves $B_n$;
(iii) there exists an invertible bounded linear, or conjugate linear, operator $A$ and a non-zero complex number $c$ such that either

$$\phi(S) = cASA^{-1} \text{ or } \phi(S) = cAS^*A^{-1}.$$
Question

It would be interesting to know if the main theorem holds true for surjective additive maps $\Phi$ on $\mathcal{L}(X)$ preserving group invertibility in one direction:

$$T \in \mathcal{G}(X) \Rightarrow \Phi(T) \in \mathcal{G}(X).$$